

dS-Holographic C -Functions with a Topological, Dilatonic Twist

by

A.J.M. Medved

*Department of Physics and Theoretical Physics Institute
University of Alberta
Edmonton, Canada T6G-2J1
[e-mail: amedved@phys.ualberta.ca]*

ABSTRACT

Recently, the holographic aspects of asymptotically de Sitter spacetimes have generated substantial literary interest. The plot continues in this paper, as we investigate a certain class of dilatonicly deformed “topological” de Sitter solutions (which were introduced in [hep-th/0110234](#)). Although such solutions possess a detrimental cosmological singularity, their interpretation from a holographic perspective remains somewhat unclear. The current focus is on the associated generalized C -functions, which are shown to maintain their usual monotonicity properties in spite of this exotic framework. These findings suggest that such topological solutions may still play a role in our understanding of quantum gravity with a positive cosmological constant.

1 Introduction

There is considerable observational evidence that the physical universe has a positive (albeit, disturbingly small) cosmological constant [1]. This observation is at least partially responsible for the recent flurry of investigations into asymptotically de Sitter spacetimes. In particular, the various holographic aspects of de Sitter (dS) space have garnered much attention. (See Ref.[2] for a list of relevant citations. Also see Refs.[3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] for more recent work.)

The focal point of dS-based holography has been a conjectured duality between asymptotically dS spacetimes and conformal field theories (CFTs) [14]. This dS/CFT duality can be viewed as an analogy to the celebrated anti-de Sitter (AdS)/CFT correspondence [15, 16, 17] (which, in turn, is an explicit realization of the renowned holographic principle [18, 19]). The CFT in a dS-inspired duality, just like its AdS analogue, lives on an asymptotic boundary of the bulk spacetime. However, contrary to the AdS case, the dS boundary is spacelike (located at temporal infinity) and the dual CFT is necessarily a Euclidean one. These distinctions can be attributed to the absence of both a globally timelike Killing vector and a spatial infinity in asymptotically dS spacetimes [20].

The various investigations into dS holography have, of course, lead to many interesting deductions and observations. At the forefront of these is Bousso’s realization of an entropic upper bound [21]. More specifically, the entropy of pure dS space serves as an upper bound on the total entropy that can be stored in any spacetime with a positive cosmological constant.¹ With guidance from the “Bousso bound”, Balasubramanian, de Boer and Minic [23] have proposed a similar upper limit on the total mass of an asymptotically dS spacetime. In particular, these authors have conjectured that any such spacetime whose conserved mass² exceeds that of pure dS space will contain a naked cosmological singularity.

For the sake of argument, let us accept the conjectured mass bound as being a true property of asymptotically dS spacetimes (as recent analysis does seem to support [26]). In this case, from a bulk viewpoint, the im-

¹Note that the validity of the Bousso bound does depend on some form of the positive energy condition [22].

²The “BdBM bound” implies a specific definition for the mass [23, 24]. This definition can be viewed as an appropriate generalization of the Brown-York quasi-local energy [25].

plications are quite severe; a violation of this bound results in a type of singularity that is considered to be *non grata* in most cosmological models. On the other hand, from a holographic perspective, the implications are somewhat murkier. As pointed out by Ghezelbash and Mann [27], for a hypothetical observer located on an asymptotic spacelike boundary (i.e., a “CFT observer”), any such singularity will remain causally hidden behind the cosmological horizon. That is to say, any quantity measured by this observer depends only on the boundary theory; in fact, the observer need not be aware that an interior region even exists. To reword this in a philosophical sense, what exactly constitutes the “true physical” picture: the boundary theory or the bulk (or both or neither)?

Given the stated ambiguity, one might argue that bound-violating asymptotically dS solutions should not be dismissed *a priori*. Such solutions have been explicitly formulated in a paper by Cai, Myung and Zhang [26] (with their original motivation being to test the mass-bound conjecture). One of these so-called “topological” de Sitter (TdS) solutions can effectively be obtained with a sign reversal (in the mass term) of a more conventional Schwarzschild-dS solution. As a consequence, the black hole horizon disappears, leaving behind a naked singularity enclosed by the usual cosmological horizon.³ Furthermore, it can readily be confirmed that, for any TdS solution, the conserved mass (in accordance with the definition of Ref.[23]) does indeed exceed that of its purely dS counterpart.

Since the original presentation by Cai *et al.* [26], some subsequent papers have considered the implications of a possible TdS/CFT duality [29, 30, 31, 2]. For the most part, this duality would appear to be preferential to its Schwarzschild-dS counterpart;⁴ inasmuch as the CFT energy can only be positive in the TdS case [32, 29]. (Significantly, a negative energy implies a non-unitary theory.) In view of this desirable feature, we argue that TdS solutions merit further investigation, and proceed on this basis.

Ultimately, one might hope that quantum gravity can be used to deduce the validity (or invalidity) of a field theory that is holographically dual to a

³It should be kept in mind that a TdS solution can have a spherical, hyperbolic or flat (cosmological) horizon geometry. This is contrary to the Schwarzschild-dS case, which only permits a spherical horizon geometry. It is this diversity in TdS horizon geometries that prompted the topological nomenclature in analogy with Ref.[28].

⁴In Ref.[31], however, the author argues against TdS solutions in a dynamical-boundary scenario. We elaborate on this point in the final section.

TdS bulk. However, as it is well known, a definitive theory of quantum gravity remains currently out of reach. Even the most promising candidate, string theory, fails to provide a suitable description of dS space [20]. Nevertheless, we propose that much can still be learned by subjecting the (conjectured) TdS/CFT correspondence to various holographic “consistency checks”.

With the above proposal in mind, let us consider the intriguing phenomena of holographic renormalization group (RG) flows. Significantly, RG flows are commonly believed to play a prominent role in any holographic bulk/boundary duality. (For instance, see Refs.[33, 34, 35, 36, 37, 23].) More specifically, as any relevant parameter of a bulk spacetime evolves, a RG flow is expected to be induced in the dually related boundary theory. This picture follows from the so-called ultraviolet/infrared correspondence [38, 39], which implies that bulk evolution towards the infrared translates into boundary flow towards the ultraviolet and *vice versa*.

What is particularly pertinent to the holographic-RG picture is the existence of a generalized C -function (in analogy with RG flows in a two-dimensional CFT context [40]). Moreover, this C -function should exhibit various monotonicity properties that are reflective of the underlying ultraviolet/infrared duality. In view of all this, an appropriate analysis of a prospective C -function should serve as a suitable “laboratory” for testing the viability of any conjectured bulk/boundary correspondence. Just such an analysis, in a TdS context, will serve as the focal point of the current paper.

If one sets out to “test” a prospective C -function, it should be significantly more informative when non-trivial matter fields are allowed in the bulk theory. (See, for instance, Ref.[41].) For this reason, we will generalize our considerations to a certain class of solutions that can be viewed as dilatonic deformations of a TdS geometry [26]. This new class can alternatively be viewed as domain wall solutions having a (flat) cosmological horizon. In fact, these solutions are essentially analytic continuations of domain wall-black hole spacetimes that effectively describe a truncated theory of gauged supergravity [42, 43, 44, 45].

Before discussing the content of the paper, let us consider a pair of caveats. Firstly, it should be kept in mind that the dilatonic deformations are, in general, **not** asymptotically de Sitter. Nonetheless, in the limit of a constant dilaton field, a TdS solution (with flat horizon geometry) will always be obtained [26]. Secondly, it is worth emphasizing that the presence of bulk matter (in this case, the dilaton) will typically break the conformal

symmetry of a dual boundary theory. That is to say, the holographic duality now under consideration can be viewed as a dS analogue [46] of the domain wall/quantum field theory (QFT) correspondence [43, 47, 44].⁵

The remainder of this paper is organized as follows. In Section 2, we begin by introducing the relevant action and formulating the $(n+2)$ -dimensional bulk solutions of interest: dilatonic deformations of a topological de Sitter spacetime with a flat cosmological horizon [26]. (We will subsequently refer to these as “DTdS” solutions.) In what is essentially a review of material in Ref.[26], we then go on to calculate the quasi-local stress tensor [25] and conserved mass of a DTdS solution. Notably, this calculation necessitates that a surface counterterm be added to the action for the purpose of regulating infrared divergences [48, 24, 49, 23, 27].

In Section 3, we consider a (presumably) dual, Euclidean QFT that lives on an asymptotic boundary of the bulk spacetime. In particular, we obtain explicit expressions for the QFT stress tensor and thermodynamics by making appropriate identifications with properties of the DTdS bulk. A Cardy-like form [50] for the QFT entropy is then verified. This result can be viewed as a generalization of the Cardy-Verlinde formula [51, 52], as appropriate for a flat horizon geometry [41].⁶ Significantly to later analysis, we also identify the Casimir entropy [51] of the QFT.

The focus of Section 4 is on prospective (generalized) C -functions in a DTdS/QFT holographic framework. Here, we consider a pair of prescriptions for the quantity of interest. The first is based on a formula that expresses the C -function in terms of local bulk geometry. (See, for instance, Refs.[34, 35].) The second follows from the premise that the Casimir entropy of a boundary theory [51] can be regarded, quite literally, as a Cardy-like “central charge” [50] and, hence, C -function [40]. Both of the prescribed forms are rigorously tested to see if they evolve monotonically with respect to variations in relevant parameters. Given a few justifiable assumptions, we are able to demonstrate that this is, indeed, always the case.

Finally, Section 5 provides a summary and further discussion; including a brief account on dynamical-boundary scenarios.

⁵Notably, the domain wall/QFT duality (assuming its validity) includes the AdS/CFT correspondence as a very special case.

⁶The Cardy-Verlinde formula has already been generalized for a multitude of holographic scenarios. Consult Refs.[53, 54] for a list of relevant citations.

2 Dilatonic Deformations of TdS Spacetime

In this section, we will begin by formulating the bulk theory of interest; namely, a dilatonic deformed “topological-de Sitter” solution with a flat horizon geometry [26]. Keep in mind that this “DTdS” solution describes a domain wall spacetime with a cosmological (but no black hole) horizon.

2.1 Domain Wall Action and Solutions

To start off, let us consider an action that describes an $n+2$ -dimensional dilaton-gravity theory with a Liouville-like potential. More specifically:

$$I = \frac{1}{16\pi G} \int_{\mathcal{M}} d^{n+2}x \sqrt{-g} \left[R - \frac{1}{2} (\partial\phi)^2 + V_o e^{-a\phi_o} \right] + I_{GH}, \quad (1)$$

where \mathcal{M} represents the manifold, G is the $n+2$ -dimensional Newton constant, and where V_o and a are to be regarded as positive constants. Note that we have also included the Gibbons-Hawking surface term, I_{GH} , which is necessary for a well-defined variational principle on the boundary of the manifold [55]. This surface term takes the form:

$$I_{GH} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^{n+1}x \sqrt{|h|} \mathcal{K}, \quad (2)$$

where \mathcal{K} is the trace of the extrinsic curvature on the boundary ($\partial\mathcal{M}$).

Interestingly, the above action (1) is known to effectively describe a truncated theory of gauged supergravity [42, 43]. Given this pedigree, V_o and a can be directly expressed in terms of \mathcal{N} and p , where \mathcal{N} is the number of Dp -branes in the originating theory [44].

A certain class of domain wall-black hole solutions (with Ricci-flat horizons) has been found for this action [45]. The associated metric and dilaton can be expressed by way of the following formalism:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + R(r)^2 dx_n^2, \quad (3)$$

$$f(r) = \frac{V_o e^{-a\phi_o} r^{2N}}{nN [N(n+2) - 1] l^{2N-2}} - \frac{mr^{1-nN}}{\sqrt{2nN(1-N)}}, \quad (4)$$

$$R(r) = \frac{r^N}{l^{N-2}}, \quad (5)$$

$$\phi(r) = \phi_o + \sqrt{2nN(1-N)} \ln(r). \quad (6)$$

In the above, ϕ_o and m are non-negative constants of integration, dx_n^2 is the dimensionless line element of an n -dimensional Ricci-flat spacetime, and l is some length parameter (insuring correct dimensionality) that will subsequently be set to unity. (Note that $m = 0$ corresponds to a purely domain wall spacetime.) Furthermore, the parameter N has been defined in accordance with:

$$a = \frac{\sqrt{2nN(1-n)}}{nN}. \quad (7)$$

In view of Eq.(4), the following bound should be imposed on N :

$$(n+2)^{-1} \leq N \leq 1. \quad (8)$$

For later convenience, let us rewrite Eq.(4) in the following form:

$$f(r) = br^{2N} - \frac{\tilde{m}}{r^{nN-1}}, \quad (9)$$

where:

$$b \equiv \frac{V_o e^{-a\phi_o}}{nN [N(n+2) - 1]}, \quad (10)$$

$$\tilde{m} \equiv \frac{m}{\sqrt{2nN(1-N)}}. \quad (11)$$

It is now quite evident that the special case of $N = 1$ describes, with a suitable renormalization of m , an $n+2$ -dimensional Schwarzschild-AdS black hole.

When $N < 1$, this class of solutions is no longer asymptotically AdS. However, one can still obtain a well-defined quasi-local stress tensor [25], provided that an appropriate surface counterterm has first been added to the action [48, 24, 49, 44]. The conserved mass (M) can be directly calculated via this stress-energy tensor, and one finds [45]:

$$M = \frac{nN\mathcal{V}_n\tilde{m}}{16\pi G}, \quad (12)$$

where \mathcal{V}_n is the volume of dx_n^2 (i.e., the dimensionless volume of the domain wall). Note that there is no vacuum ($m = 0$) contribution to the mass by virtue of the locally flat solution space [49].

Now let us suppose that $V_o < 0$. In this case, the action (1) can still be effectively viewed as the truncation of a gauged supergravity theory [46]. Let us further assume that m (or \tilde{m}) ≤ 0 and then, for sake of convenience, redefine $V_o \rightarrow -V_o$ and $\tilde{m} \rightarrow -\tilde{m}$. The prior solution remains unchanged except for the metric function $f(r)$, which should now be revised as follows:

$$f(r) = \frac{\tilde{m}}{r^{nN-1}} - br^{2N}. \quad (13)$$

For the special case of $N = 1$, the revised solution describes a topological de Sitter spacetime [26] with a flat (cosmological) horizon geometry. For general N (but satisfying Eq.(8)), this dilatonicly deformed solution is no longer asymptotically dS, but it does, in fact, still possess a cosmological horizon. Denoting the horizon location by $r = r_c$, we have:

$$r_c = \left[\frac{\tilde{m}}{b} \right]^{\frac{1}{N(2+n)-1}}. \quad (14)$$

It should be kept in mind that, for any allowed N , the cosmological horizon encloses a naked singularity.

The associated thermodynamics of any cosmological horizon can be obtained via standard identifications [56]. For a DTdS solution in particular, the horizon temperature and entropy are respectively calculated as follows:

$$T_H = -\frac{1}{4\pi} \left. \frac{df}{dr} \right|_{r=r_c} = \frac{b[N(n+2)-1]}{4\pi} r_c^{2N-1}, \quad (15)$$

$$S_H = \frac{"area"}{4G} = \frac{\mathcal{V}_n r_c^{nN}}{4G}. \quad (16)$$

2.2 Quasi-Local Stress Tensor and Mass

With the above thermodynamic identities, it can readily be shown that the first law of horizon thermodynamics, $dM = T_H dS_H$, is uniquely satisfied (up to the usual constant) for M as given by Eq.(12). However, it is still instructive to derive M on a more fundamental level; namely, as the conserved charge associated with time translation. Note that the following analysis essentially reviews a calculation that is found in Ref.[26].

Let us begin here by considering the spacetime outside of the cosmological horizon (i.e., $r > r_c$) and, thus, appropriately shielded from the naked

singularity. (Ultimately, we are interested in the asymptotic limit of $r \rightarrow \infty$; that is, future spacelike infinity or \mathcal{I}_+ [20].) The coordinates r and t change their character in crossing over the horizon (from spacelike to timelike and *vice versa*), and so, for illustrative purposes, we will relabel these as $r \rightarrow \tau$ and $t \rightarrow \rho$. Given that $\tau \geq \tau_c = r_c$, the DTdS metric (3,13) takes on the following suggestive form:

$$ds^2 = -f^{-1}(\tau)d\tau^2 + f(\tau)d\rho^2 + R(\tau)^2 dx_n^2, \quad (17)$$

$$f(\tau) = b\tau^{2N} - \frac{\tilde{m}}{\tau^{nN-1}} \geq 0. \quad (18)$$

Being somewhat more specific, we now focus on an $n+1$ -dimensional spacelike boundary ($\partial\mathcal{M}$). The boundary geometry can suitably be described by the following metric:⁷

$$ds_B^2 = h_{ab}dx^a dx^b = f(\tau)d\rho^2 + R(\tau)^2 dx_n^2, \quad (19)$$

where, for any specific boundary, τ is fixed at some value greater than τ_c .

For a calculation of conserved charges (via a generalized Brown-York treatment [25]), it is necessary to add an appropriate surface counterterm to the action in question [24, 49, 23, 27]. (The premise being that the counterterm can be used to cancel off the infrared divergences of a given gravitational action [48].) Generally speaking, such a counterterm is a necessarily complicated expression involving the intrinsic curvature of the boundary metric. However, for the current investigation, this intrinsic curvature vanishes (by virtue of dx_n^2 being Ricci flat) and the counterterm Lagrangian should reduce into a relatively simple, single-term form. In analogy with the deformed AdS-Schwarzschild theory [44], a suitably defined counterterm has already been identified for the DTdS form of the action [26]. This result can be expressed as follows:

$$I_{ct} = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^n x d\rho \sqrt{h} \frac{n}{L}, \quad (20)$$

where we have defined:

$$L^{-1} \equiv N\sqrt{b}\tau^{N-1}. \quad (21)$$

⁷Note the following conventions. Greek indices will imply coordinates of the $n+2$ -dimensional manifold, “low-order” Roman indices (a, b, \dots) will imply boundary coordinates, and higher-order Roman indices (i, j, \dots) will imply coordinates on the dx_n^2 hypersurface.

Here, L can be viewed as the DTdS generalization of a dS curvature radius. Although L generally varies throughout the manifold due to the factor of τ^{N-1} ,⁸ the analogy with dS space becomes evident when $N = 1$. In this limiting case, I_{ct} clearly reduces to the anticipated asymptotically dS form [23].

One can calculate the (so-called) quasi-local stress tensor [25] by varying the total action, $I + I_{ct}$ (including the Gibbons-Hawking surface term [55]), with respect to the boundary metric [24, 23]. Applying this prescription, we find:

$$T_{ab} = -\frac{2}{\sqrt{h}} \frac{\delta(I + I_{ct})}{\delta h^{ab}} = \frac{1}{8\pi G} \left[\mathcal{K}_{ab} - \mathcal{K} h_{ab} + \frac{nN b^{\frac{1}{2}}}{\tau^{N-1}} h_{ab} \right]. \quad (22)$$

To compute the extrinsic curvature (and its trace), we will employ a standard definition: $\mathcal{K}_{ab} = h_a^\mu h_b^\nu \nabla_\mu \eta_\nu$, where η^μ is the unit normal vector to the boundary. With the normal suitably taken to be $\eta^\mu = f^{\frac{1}{2}} \delta_\tau^\mu$, some straightforward calculation yields:

$$\mathcal{K}_{ij} = \delta_{ij} N f^{\frac{1}{2}} \tau^{2N-1}, \quad (23)$$

$$\mathcal{K}_{\rho\rho} = \frac{1}{2} f' f^{\frac{1}{2}}, \quad (24)$$

$$\mathcal{K} = \mathcal{K}_a^a = \frac{nN f^{\frac{1}{2}}}{\tau} + \frac{1}{2} \frac{f'}{f^{\frac{1}{2}}}. \quad (25)$$

Note that a prime indicates differentiation with respect to τ .

Substituting the above results into the stress tensor (22) and considering the large τ limit, we eventually obtain the following leading-order relations:

$$T_{ij} = -\delta_{ij} \frac{(2N-1) \tilde{m} \tau^{-(n-1)N}}{16\pi G b^{\frac{1}{2}}}, \quad (26)$$

$$T_{\rho\rho} = \frac{nN \tilde{m} b^{\frac{1}{2}} \tau^{-(n-1)N}}{16\pi G}. \quad (27)$$

⁸Actually, the factor of τ^{N-1} was omitted in the original presentation [26]. Such a factor is, however, necessary so as to ensure the correct dimensionality of the counterterm Lagrangian. Other possibilities, such as l^{N-1} (where l is the length parameter that has been set to unity), do not achieve the desired cancellation of infrared divergences.

Let us now focus our attentions on the calculation of the conserved mass. Once again, we will utilize the techniques of Refs.[24, 23], which constitute a generalization of the Brown-York methodology [25].

First of all, let us discuss the general formalism for an arbitrary $(n+2)$ -dimensional) spacetime. It is appropriate to consider an $n+1$ -dimensional boundary $(\partial\mathcal{M})$ and an n -dimensional spacelike surface, Σ , that is enclosed within $\partial\mathcal{M}$. Ideally, given a Killing vector (k^a) that generates an isometry of the boundary geometry, one would like to calculate the associated conserved charge (Q) . This task can be accomplished with the following formula [24]:

$$Q = \oint_{\Sigma} d^n\phi \sqrt{\sigma} u^a k^b T_{ab}, \quad (28)$$

where the metric on Σ has been parametrized according to $ds_{\Sigma}^2 = \sigma_{ij}\phi^i\phi^j$, u^a is the unit normal vector to Σ , and T_{ab} is the quasi-local stress tensor associated with $\partial\mathcal{M}$.

Next, let us specialize to an asymptotically dS spacetime and a calculation of the conserved mass (i.e., $Q = M$). Normally, one associates the conserved mass with a globally timelike Killing vector. Although there is no such Killing vector for an asymptotically dS spacetime, it has satisfactorily been shown that one can consider the analytic continuation of ∂_t [23]; where t is the temporal coordinate inside of the cosmological horizon. If we take ρ as being the analytic continuation of t (outside of the cosmological horizon) and conveniently choose u^a to be proportional to the relevant Killing vector, then Eq.(28) takes on the following form:

$$M = \oint_{\Sigma} d^n\phi \sqrt{\sigma} N_{\rho} u^{\rho} u^{\rho} T_{\rho\rho}, \quad (29)$$

where the Killing vector has been normalized according to $k^a = N_{\rho} u^a$. (N_{ρ} can be identified with the usual “lapse function” of the boundary metric.)

Finally, we are in a position to consider the model currently under investigation. Assuming that the above formalism can directly be adapted into a DTdS framework,⁹ we can accordingly re-express Eq.(29) as follows:

$$M = \mathcal{V}_n \tau^{nN} f^{-\frac{1}{2}}(\tau) T_{\rho\rho}. \quad (30)$$

⁹It is worth repeating that DTdS solutions are not asymptotically dS, except in the special case of $N = 1$.

To obtain this form, the following identities have been applied: $\oint_{\Sigma} d^n \phi \sqrt{\sigma} = \mathcal{V}_n \tau^{nN}$, $u^a = \delta^a_{\rho} f^{-1/2}$, and $N_{\rho} = f^{1/2}$.

Substituting Eq.(27) into the above relation and taking the asymptotic limit, we ultimately find:

$$\lim_{\tau \rightarrow \infty} M = \frac{nN}{16\pi G} \mathcal{V}_n \tilde{m}. \quad (31)$$

Fortunately, this outcome agrees with the priorly quoted result (12) and, thus, with the first law of DTdS horizon thermodynamics.

3 Euclidean QFTs on the Boundary

It has been suggested that the AdS/CFT correspondence [15, 16, 17] is really just a special case of a more general holographic duality; namely, the domain wall/QFT correspondence [43, 47, 44]. By way of analogy, one might argue that the dS/CFT correspondence [14] is also a special case of a more encompassing duality. That is to say, there may exist a dual relationship between certain domain wall spacetimes and Euclidean QFTs [46, 26]. Although somewhat speculative, we will adopt this viewpoint for the duration of the paper.

3.1 QFT Geometry and Stress Tensor

With the above discussion in mind, let us now consider a Euclidean QFT that lives on an asymptotic boundary of a DTdS bulk spacetime. Presumably (or perhaps naively), the bulk and boundary theories should be dually related in a holographic sense. Keep in mind that, for the very special case of $N = 1$, the (TdS) bulk theory is asymptotically dS, the boundary theory is a conformal one, and a holographic duality appears to be in evidence [29, 30, 31, 2].

As is typically the case in holographic bulk/boundary dualities, the metric of the QFT in question will be fixed, up to a conformal factor, as the metric on an asymptotic boundary of the bulk spacetime. By analogy with Ref.[52], we invoke:

$$\begin{aligned} ds^2_{QFT} &= \gamma_{ab} dx^a dx^b \\ &= \lim_{\tau \rightarrow \infty} \frac{1}{\tau^{2N} b} ds^2 \end{aligned}$$

$$= d\rho^2 + b^{-1}dx_n^2, \quad (32)$$

where ds^2 (in the second line) is the metric defined by Eqs.(17,18). Let us re-emphasize that all current/future considerations are restricted to the region outside of the cosmological horizon (i.e., $\tau \geq \tau_c$).

We can calculate the stress tensor (\mathcal{T}_{ab}) of the QFT by way of the following relation [57]:

$$\sqrt{\gamma}\gamma^{ab}\mathcal{T}_{bc} = \lim_{\tau \rightarrow \infty} \sqrt{h}h^{ab}T_{bc}, \quad (33)$$

where T_{ab} is the quasi-local stress tensor of Eqs.(26,27) and the boundary metric, h_{ab} , is defined by Eq.(19).

Utilizing the above relation, we are able to deduce:

$$\mathcal{T}_{ij} = -\delta_{ij} \frac{(2N-1)\tilde{m}b^{\frac{n-2}{2}}}{16\pi G}, \quad (34)$$

$$\mathcal{T}_{\rho\rho} = \frac{nN\tilde{m}b^{\frac{n}{2}}}{16\pi G}. \quad (35)$$

It is interesting to note that the trace of the stress tensor, $\mathcal{T} = \mathcal{T}_a^a$, only vanishes in the special case of $N = 1$. Reassuringly, this special case describes an asymptotically dS spacetime, and so the corresponding QFT should, indeed, be a conformal theory. Of further interest, the above results imply the following equation of state for the QFT (with ϵ and p respectively denoting energy density and pressure):

$$\begin{aligned} \omega \equiv \frac{p}{\epsilon} &= -b \frac{\mathcal{T}_{ij}\delta_j^i}{\mathcal{T}_{\rho\rho}} \\ &= \frac{2N-1}{nN}, \end{aligned} \quad (36)$$

where we have assumed a perfect-fluid description. As expected, this equation reduces to $\omega = 1/n$ (i.e., radiative matter) when the $N = 1$ conformal theory is realized.

3.2 QFT Thermodynamics

Let us now consider the thermodynamic properties of this QFT and subsequently determine if they can accommodate a Cardy-Verlinde-like entropic

form [50, 51]. As is the usual practice, we first rescale the QFT metric so that the associated boundary is located at a fixed “radial distance”, \mathcal{R} , from the origin.¹⁰ This necessitates the following conformal transformation:

$$ds_{QFT}^2 \rightarrow b\mathcal{R}^2 ds_{QFT}^2 = b\mathcal{R}^2 d\rho^2 + \mathcal{R}^2 dx_n^2. \quad (37)$$

However, we have not yet realized the desired form; that is:

$$ds_{QFT}^2 = d\rho^2 + \mathcal{R}^2 dx_n^2. \quad (38)$$

Evidently, the Euclidean time coordinate should be further rescaled such that $\rho \rightarrow b^{1/2}\mathcal{R}\rho$. It follows that, from a QFT perspective, the bulk energy and temperature (having units of inverse time) should be “red shifted” by a factor of $\Delta = [b^{1/2}\mathcal{R}]^{-1}$.

In accordance with the above discussion, the QFT thermodynamics can be identified as follows:¹¹

$$\begin{aligned} E_{QFT} = \Delta M &= \frac{nN\mathcal{V}_n\tilde{m}}{16\pi G b^{\frac{1}{2}}\mathcal{R}} \\ &= \frac{nN\mathcal{V}_n b^{\frac{1}{2}}}{16\pi G \mathcal{R}} \tau_c^{N(2+n)-1}, \end{aligned} \quad (39)$$

$$T_{QFT} = \Delta T_H = \frac{b^{\frac{1}{2}}[N(n+2)-1]}{4\pi\mathcal{R}} \tau_c^{2N-1}, \quad (40)$$

$$S_{QFT} = S_H = \frac{\mathcal{V}_n}{4G} \tau_c^{nN}, \quad (41)$$

where we have incorporated Eqs.(14,15,16,31). It can readily be verified that these relations satisfy the first law of QFT thermodynamics; that is, $dE_{QFT} = T_{QFT}dS_{QFT}$. We also take note of $E_{QFT} \geq 0$; which is indicative of a topological-dS, rather than Schwarzschild-dS, holographic framework [32, 29].

In general and regardless of dimensionality, the entropy of any horizon and, by duality, the entropy of its holographic boundary theory should be

¹⁰Strictly speaking, \mathcal{R} represents temporal evolution when outside of the cosmological horizon.

¹¹Note that the entropy is universally unaffected by any such coordinate rescaling [17].

expressible in a Cardy-like form [58, 59]. That is to say, one might expect [50]:

$$S_{QFT} = \frac{2\pi}{n} \sqrt{\frac{c}{6} \left[L_o - \frac{c}{24} \right]}, \quad (42)$$

where $L_o = \mathcal{R}E_{QFT}$ and the “central charge”, c , is directly proportional to the Casimir (i.e., sub-extensive) energy of the boundary theory [51]. However, given a bulk theory with a flat horizon geometry (which implies a vanishing Casimir energy), the associated QFT can be expected to conform with the following version [41, 54]:

$$S_{QFT} = \frac{2\pi}{n} \sqrt{\frac{cL_o}{6}}. \quad (43)$$

As a consequence of this form, c is, in principle, proportional to an appropriately defined Casimir entropy [51] (which, unlike the Casimir energy, remains finite and positive, regardless of the horizon geometry [41, 60]).

We find that the above thermodynamic relations do indeed satisfy Eq.(43) (with $L_o = \mathcal{R}E_{QFT}$), as long as:

$$c = \frac{3n\mathcal{V}_n}{2\pi G N b^{\frac{1}{2}}} \tau_c^{N(n-2)+1}. \quad (44)$$

The expected relation between the Casimir entropy (S_C) and the generalized central charge is $S_C = \pi c/6n$ (for instance, [41]). On this basis:

$$S_C = \frac{\mathcal{V}_n}{4GNb^{\frac{1}{2}}} \tau_c^{N(n-2)+1}. \quad (45)$$

It is hard to confirm the validity of this result, insofar as we are unable to calculate the Casimir entropy by more direct means. (This is contrary to the usual scenario for a spherical horizon: the Casimir entropy is directly proportional to the Casimir energy, which represents a violation in the Euler identity [51].) Nonetheless, it is quite reassuring that S_C (as defined above) reduces to its anticipated form in the special $N = 1$ case [30].

4 Generalized C-Functions

With inspiration from the ultraviolet/infrared correspondence [38, 39], it is commonly believed that evolution of a bulk spacetime will give rise to some

form of RG flow in its holographically related boundary theory (for instance, [33, 34, 35, 36, 37, 23]). Moreover, there should exist some generalized C -function (in analogy with two-dimensional RG flows [40]) that exhibits appropriate monotonicity properties as the state of the system varies. On the basis of such arguments, the existence (or lack thereof) of a suitable C -function should serve as an appropriate litmus test for a conjectured bulk/boundary duality. This philosophy, in a DTdS/QFT context, will serve as the premise for the analysis that follows.

Given a bulk spacetime and its dually related boundary theory, there are two commonly used prescriptions for the generalized C -function. We will examine both of these in turn.

4.1 Bulk-Geometrical Prescription

Firstly, let us consider any bulk ($n+2$ -dimensional) spacetime for which the metric can be expressed in the following domain wall-like form:

$$ds^2 = -dz^2 + e^{2A(z)} [dy^2 + dx_n^2]. \quad (46)$$

In this case, one expects the existence of a generalized C -function that is based on local bulk geometry and can be represented as follows (for instance, [34, 35]):¹²

$$C \sim \frac{1}{G [A'(z)]^n}. \quad (47)$$

Note that a prime now indicates differentiation with respect to z .

The DTdS bulk metric of Eq.(17) can be cast into the above template (46) by way of the following identifications:

$$dz = \frac{1}{\sqrt{f(\tau)}} d\tau, \quad (48)$$

$$y = \frac{\sqrt{f(\tau)}}{R(\tau)} \rho, \quad (49)$$

¹²A very recent paper [11] considered a revised form for this C -function, which apparently has a wider range of applicability. However, in the case of a flat horizon geometry (as is relevant to the current study), this newer formulation reduces to Eq.(47).

$$A(z) = A(\tau) = \ln[R(\tau)] = N \ln(\tau). \quad (50)$$

As it stands, such a calculation of C would not be particularly enlightening. Nevertheless, we can still viably proceed by first assuming that the effective mass parameter, \tilde{m} , is much smaller than the other relevant scales. (However, \tilde{m} should remain a non-vanishing quantity, so that a cosmological horizon is still in existence.) It is significant that, with this assumption, $f(\tau) \approx b\tau^{2N}$ becomes a valid approximation (cf. Eq.(18)).

Applying the pertinent approximation to Eq.(48), we obtain the following useful relation:

$$dz \approx \frac{\tau^{-N}}{b^{\frac{1}{2}}} d\tau. \quad (51)$$

In terms of the above formalism, the prescribed C -function (47) now yields:

$$C \sim \frac{\tau^{n(1-N)}}{GN^n b^{\frac{n}{2}}}. \quad (52)$$

It is immediately clear that C increases monotonically with increasing “radial distance” τ (recalling that $N \leq 1$). Hence, we have confirmed the anticipated ultraviolet/infrared duality [38, 39]. That is, the infrared (large τ) limit of the bulk theory corresponds to the ultraviolet (large C) limit of the QFT and *vice versa*.

Interestingly, we see that C becomes a constant (with respect to variations in τ) when $N = 1$. That is, the constant-dilaton (TdS) theory translates to a conformal fixed point of the holographic RG flow. Clearly, this $N = 1$ fixed point is an infrared one.

It should also be instructive to examine the behavior of the C -function under variations in N . Significantly, changes in N reflect variations in the matter content of the theory. To put it another way, as N monotonically decreases below its conformal value of 1, the bulk scalar fields are effectively being “turned on” (cf. Eq.(6)). In fact, simple analysis tells us that the dilaton field will continue to “grow” until $N = 1/2$ has been reached. Thus, one might expect $N = 1/2$ to represent a fixed point in the associated RG flow. (Clearly, the conformal value, $N = 1$, serves as the other fixed point.) We will provide further support for this claim in the latter part of this section.

To proceed along the suggested line, it is necessary to re-express Eq.(52) so that all N dependence is explicit. We can accomplish this task by substi-

tuting for $b = b(N)$ via Eq.(10). This process yields:

$$C \sim \frac{\tau^{n(1-N)}}{G} \left[\frac{nN(n+2) - n}{NV_o} \right]^{\frac{n}{2}}, \quad (53)$$

where we have set $\phi_o = 0$ for sake of convenience.

Given that C is strictly a positive quantity, it is just as informative (and substantially easier) to consider variations in $\ln(C)$. Hence, it is useful to write:

$$\ln(C) = n(1-N) \ln(\tau) + \frac{n}{2} \ln[N(n+2) - 1] - \frac{n}{2} \ln(N) + \text{constant}. \quad (54)$$

Varying this expression with respect to N , we have:

$$\frac{\partial \ln(C)}{\partial N} = -\ln(\tau) + \frac{n}{2N} [N(n+2) - 1]^{-1}. \quad (55)$$

The above result indicates that, for sufficiently large values of τ , C is a monotonically decreasing function of N . Furthermore, since 1 is an upper bound on N , the conformal theory can be identified as the infrared fixed point with respect to bulk-matter evolution. Given that the $N = 1$ limit corresponds to an essentially matter-free theory, this identification seems pleasantly intuitive. At this juncture, however, the ultraviolet fixed point seems somewhat less clear.

Before proceeding on to the next phase of the analysis, let us comment on the condition of “sufficiently large τ ”. This constraint can be viewed as a manifestation of a certain aspect of the DTdS/QFT framework. In particular, any external observer will be unable to access information from behind the cosmological horizon and the duality must, therefore, naturally break down when $\tau \leq \tau_c$. To put it another way, τ_c can be viewed as a necessary ultraviolet cutoff for the DTdS bulk or (by duality [38, 39]) an infrared cutoff for the QFT.

4.2 Casimir-Entropic Prescription

Alternatively, given a holographic boundary theory, the Casimir entropy (S_C) has also been interpreted as a generalized C -function. (With considerable success; see, for instance, Refs.[41, 61]). This interpretation of S_C follows

directly from its role as an effective central charge [50, 40] in the Cardy-Verlinde formula [51]. Recalling Section 3, we have already identified the Casimir entropy (45) for the QFT of interest. On this basis, let us now consider:

$$C = S_C = \frac{\mathcal{V}_n}{4GNb^{\frac{1}{2}}} \tau_c^{N(n-2)+1}. \quad (56)$$

Considering our “gameplan”, this C -function can most conveniently be expressed as an explicit function of T_{QFT} and N . Applying Eq.(10) for $b = b(N)$ and Eq.(40) for the temperature, we eventually obtain the following expression:

$$C = \frac{\mathcal{V}_n}{4G} \left[\frac{n}{V_o} \right]^{\frac{nN}{2(2N-1)}} \left[\frac{N(n+2)-1}{N} \right]^{\frac{N(4-n)-2}{2(2N-1)}} [4\pi\mathcal{R}T]^{\frac{N(n-2)+1}{2N-1}}. \quad (57)$$

Note that ϕ_o has again been set to vanish and the subscript “QFT” has been dropped from the temperature.

As discussed before, it is both convenient and sufficient to consider the logarithm of C . Up to some irrelevant constant terms, we find the following:

$$\begin{aligned} \ln(C) = & -\frac{nN}{2(2N-1)} \ln\left(\frac{V_o}{n}\right) + \frac{N(4-n)-2}{2(2N-1)} \ln\left[\frac{N(n+2)-1}{N}\right] \\ & + \frac{N(n-2)+1}{2N-1} \ln(4\pi\mathcal{R}T). \end{aligned} \quad (58)$$

Let us first consider varying $\ln(C)$ with respect to the boundary radius (\mathcal{R}):

$$\mathcal{R} \frac{\partial \ln(C)}{\partial \mathcal{R}} = \frac{N(n-2)+1}{2N-1}. \quad (59)$$

If we assume that (i) $n \geq 2 - (1/N)$ and (ii) $N \geq \frac{1}{2}$, then the following bound can be established:

$$\frac{\partial \ln(C)}{\partial \mathcal{R}} \geq 0. \quad (60)$$

That is, the ultraviolet/infrared connection has (once again) been verified. Some commentary on the assumed conditions is, however, still in order.

Condition (i) simply limits considerations to a bulk theory of dimensionality three or greater, which obviously covers all physically relevant dimensionalities.

Condition (ii) is more difficult to interpret, given that, strictly speaking, N can take on values as low as $(n+2)^{-1}$ (cf. Eq.(8)). Nonetheless, it is also of relevance that any value of N below $\frac{1}{2}$ translates into a QFT with a negative pressure; cf. Eq.(36). Interestingly, it has been argued that the domain wall/QFT correspondence will break down for these types of negative-pressure states [44, 45]. (This argument follows from an observation that gravity fails to decouple from the QFT when the pressure falls below zero.) In view of this consideration, $N \geq 1/2$ seems to be quite a natural constraint. Furthermore, this lower bound on N supports a prior hypothesis; namely, that $N = 1/2$ is the most suitable candidate for an ultraviolet fixed point. To reiterate, this hypothesis is based on the observance that, as N decreases below 1, the dilaton will grow until $N = 1/2$ has been reached.

Next, let us consider how the C -function evolves when the temperature is varied. It is clear that C has the same functional dependence on T as it has on \mathcal{R} . Hence, imposing the same justifiable constraints as before, we have:

$$\frac{\partial \ln(C)}{\partial T} \geq 0. \quad (61)$$

That is, the C -function evolves monotonically with respect to temperature and, moreover, the QFT flows towards the ultraviolet as temperature increases. This outcome agrees with the usual expectation that thermal excitations will induce additional degrees of freedom.

Finally, let us consider variations in the C -function with respect to the parameter N ; keeping in mind that a decreasing N translates into excitations of the bulk scalar. By way of Eq.(58), the following is found:

$$\begin{aligned} \frac{\partial \ln(C)}{\partial N} = & - \frac{n}{(2N-1)^2} \left[\ln(4\pi\mathcal{R}T) - \frac{1}{2} \ln \left(\frac{N(n+2)-1}{nNV_o^{-1}} \right) \right] \\ & - \frac{N(n-4)+2}{2N(2N-1)} [N(n+2)-1]^{-1}. \end{aligned} \quad (62)$$

It is evident that, for both “sufficiently large” values of temperature and $N \geq \frac{1}{2}$, the following relation will always be satisfied:

$$\frac{\partial \ln(C)}{\partial [-N]} \geq 0. \quad (63)$$

Hence, under suitable conditions, C is a monotonically increasing function as N decreases. This result agrees with our previous finding; thus reconfirming the intuitive notion of bulk matter fields inducing a flow to the ultraviolet.

Let us now comment on the most recently imposed conditions. $N \geq \frac{1}{2}$ is just the previously discussed positive-pressure constraint, whereas the condition of large temperature can be justified by virtue of the following argument. The Cardy-Verlinde formula, upon which this definition of C has been based, only has validity in a regime of large temperature [51]. In fact, a breakdown can be expected in the Cardy-Verlinde formalism when $\mathcal{R}T \gg 1$ is no longer satisfied [62, 63, 64]. It is clear that the same limitation can be deduced from our findings.

As an aside, it would be interesting to determine if there is some finite temperature at which Eq.(63) does indeed begin to fail. Significantly, this special value of temperature could be interpreted as the analogue of the Hawking-Page (Schwarzschild-AdS) phase transition [65]. However, it appears that such a determination would require a complicated numerical analysis.

To briefly summarize, we have demonstrated that both definitions of the generalized C -function (47,56) satisfy the monotonicity properties that would be expected for a bulk spacetime with a QFT dual.

5 Concluding Discussion

In summary, we have been investigating into the holographic properties of a special class of domain wall solutions; these having the distinction of a singularity enclosed by a cosmological horizon. Any of these bulk solutions can be interpreted as a dilatonic deformation of a topological de Sitter spacetime [26]. That is to say, in the limit of a constant dilaton field, the once-deformed model will describe an asymptotically de Sitter solution with a cosmological singularity. As discussed in Section 1, such a singularity may not be a particularly critical issue from the perspective of an asymptotic boundary observer [27]. Also of note, any of these deformed “DTdS” solutions can be analytically continued into a domain wall spacetime that effectively describes a truncated theory of gauged supergravity [42, 43, 44].

The initial phase of the analysis entailed strictly bulk considerations. We began here by introducing the relevant (arbitrary-dimensional) action, which

describes gravity coupled to a dilaton field with a Liouville-like potential. For this action, a certain class of domain wall-black hole solutions are known [45], whereby a trivial redefinition of the potential leads to a DTdS solution space. With the bulk geometry precisely formulated, we went on to calculate the quasi-local stress tensor and conserved mass [25] by way of the surface-counterterm method [48, 24, 49, 23]. Significantly, the calculated mass is precisely that which satisfies the first law of cosmological horizon thermodynamics. (We again note that this portion of the paper was essentially a review of prior work [26].)

In the second phase of the analysis, we considered a Euclidean quantum field theory that lives on an asymptotic boundary of the DTdS bulk space-time. It was argued that the bulk and boundary theories could well have a dual relationship [46] in analogy with the domain wall/QFT correspondence [43, 47, 44]. Utilizing standard holographic relations [57, 51, 52], we were able to identify the stress tensor and thermodynamic properties of this QFT. It was then shown that the QFT entropy satisfies a generalized form of the Cardy-Verlinde formula [50, 51]. Notably, this generalization can be viewed as the appropriate one for a flat horizon geometry [41, 54]. On the basis of this formulation, we also identified the Casimir (i.e., sub-extensive) entropy of the QFT.

The final phase of the analysis focused on generalized C -functions. In this regard, we studied two commonly used prescriptions: (i) a formula that expresses C in terms of local bulk geometry (for instance, [34, 35]) and (ii) an identification between C and a QFT-induced Casimir entropy (for instance, [41, 61]). After formulating the appropriate expressions, we tested these prescriptions by varying each of the C -functions with respect to relevant parameters. For both versions, the ultraviolet/infrared correspondence [38, 39] was clearly established. (That is, C increases monotonically with respect to an increasing boundary radius.) Furthermore, after imposing a few justifiable conditions, we were able to show that C increases monotonically with respect to both increasing temperature and decreasing N ; the latter being a parameter of the bulk theory.¹³ Both of these outcomes agree with prior expectations, given that thermal excitations should activate degrees of freedom and a decrease in N can be correlated with the activation of bulk

¹³For sake of accuracy, let us note that the temperature correspondence was only verified with the second prescription, whereas the N correspondence was verified for both.

matter fields.

Also of interest, a pair of fixed points was identified for the implied renormalization group flows. In particular, the conformal (or constant dilaton) case of $N = 1$ corresponds to an infrared fixed point, while $N = 1/2$ describes an ultraviolet fixed point. The significance of the ultraviolet limit is that it describes a boundary theory with vanishing pressure; at which point, a breakdown can be expected in the (prospective) DTdS/QFT duality [44, 45].

In conclusion, the results of our analysis are definitely in support of a DTdS/QFT and (hence) TdS/CFT correspondence. Further support for the latter duality has come in prior studies; most notably, it has been observed that a TdS bulk gives rise to a positive-energy CFT [29, 2] (in direct contrast to the more conventional Schwarzschild-dS bulk scenario [32]). However, we do not mean to imply that these studies in any way verify the legitimacy of topological de Sitter solutions (i.e., asymptotically dS solutions with a cosmological singularity). Rather, we view these positive outcomes as an argument that such solutions should not be disregarded *a priori*. Let us remind the reader that an asymptotic boundary observer would be causally inhibited from accessing any information from behind the TdS cosmological horizon; including information about the naked singularity [27]. Hence, the potential legitimacy of TdS solutions seems to depend on what constitutes the “fundamental” theory; the bulk or the boundary. It appears that this issue necessitates further investigation.

As a final consideration, we will provide a somewhat brief account on dynamical-brane scenarios.¹⁴ In a prior paper [30], the implications of a brane that evolves in a TdS background spacetime were considered. More specifically, the methodology of Savonije and Verlinde [52], for a radiation-dominated brane universe moving in a Schwarzschild-AdS bulk, was suitably generalized for a TdS framework.¹⁵ Remarkably, the desirable features of the Savonije-Verlinde treatment do indeed persist for a TdS bulk. (These features include various holographic entropy bounds [51], the saturation of these bounds when the brane crosses the bulk horizon, and the coincidence of the FRW equations¹⁶ with CFT thermodynamics at this saturation point.)

¹⁴In this aside, we prefer brane in lieu of boundary for no particular reason.

¹⁵Note that the Savonije-Verlinde program was first generalized for an asymptotically dS bulk by Ogushi [66].

¹⁶FRW implies the Friedman-Robertson-Walker cosmological equations, which typically describe brane dynamics.

However, it has subsequently been noted by Myung [31] that, for a TdS bulk, the induced brane metric and (hence) Hubble parameter are defined with respect to Euclidean cosmological time. When one continues back to Lorentzian cosmological time, the FRW equations are then expressed in terms of an undesirable negative energy density. This problem could be circumvented by adding suitable matter to the brane; however, this is just the type of fine-tuning mechanism that one usually tries to avoid.

If, for sake of argument, the above criticism is disregarded, then it is not difficult to further generalize the Savonije-Verlinde program (and its pertinent outcomes) [52] to the case of a dilatonicly deformed TdS bulk space-time. One needs only to follow the procedure outlined by Cai and Zhang [45] for the analogous case of a “dilatonicly deformed Schwarzschild-AdS” bulk spacetime (i.e., the spacetime described by Eqs.(3-6)). The only significant difference would be in the definition of the cosmological time parameter (η). More precisely, Eq.(3.4) from their paper:¹⁷

$$f(r) \left(\frac{dt}{d\eta} \right)^2 - \frac{1}{f(r)} \left(\frac{dr}{d\eta} \right)^2 = 1 \quad f(r) = br^{2N} - \frac{\tilde{m}}{r^{nN-1}}, \quad (64)$$

which leads to an induced brane metric of:

$$ds^2 = -d\eta^2 + \mathcal{R}^2(\eta)dx_n^2, \quad (65)$$

should be replaced with:

$$f(\tau) \left(\frac{d\rho}{d\eta} \right)^2 - \frac{1}{f(\tau)} \left(\frac{d\tau}{d\eta} \right)^2 = 1 \quad f(\tau) = b\tau^{2N} - \frac{\tilde{m}}{\tau^{nN-1}}, \quad (66)$$

which then leads to the following Euclidean form:

$$ds^2 = d\eta^2 + \mathcal{R}^2(\eta)dx_n^2. \quad (67)$$

(Note that $\tau \geq \tau_c$ is mandatory, whereas any $r \geq 0$ is allowed.) After making this substitution, one can essentially follow Ref.[45] in a straightforward manner. Their results (which, for the most part, agree with those of Savonije and Verlinde [52]) will persist unfettered for the DTdS case.¹⁸

¹⁷We have slightly modified the notation of Ref.[45] for the purpose of (hopefully) avoiding confusion.

¹⁸However, one should keep in mind that, for the DTdS case, the Hubble constant is Euclidean (rather than Lorentzian) and the bulk horizon is a cosmological one (rather than a black hole).

Finally, let us comment on one of the more interesting outcomes of Ref.[45], which applies vicariously to the deformed TdS model. As it so happens, the resultant FRW equations imply the following equation of state on the brane:

$$\omega = \frac{p}{\epsilon} = \frac{1}{nN}. \quad (68)$$

This only agrees with our prior finding (*viz.* Eq.(36)):

$$\omega = \frac{2N - 1}{nN} \quad (69)$$

in the special conformal case of $N = 1$. As argued by Cai and Zhang, this discrepancy can be attributed to an *effective* gravitational constant that varies in time [45]:

$$G_{n+1} = \frac{(n-1)Nb}{[\mathcal{R}(\eta)]^{\frac{1-N}{N}}} G, \quad (70)$$

as well as the contributions from a dynamical dilaton field.¹⁹

The dynamical contributions from gravity and the dilaton also seem to compensate for the effects of the apparently non-radiative matter (since, in general, $\omega \neq n^{-1}$). That is to say, the holographic bounds and brane-horizon coincidences observed by Savonije and Verlinde [52] can only be expected to persevere in a radiation-dominated universe [67]; so it seems strange that, for the most part, they do so here. We hope to investigate this matter in a future work.

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References

- [1] See, for instance, N. Bahcall, J.P. Ostriker, S. Perlmutter and P.J. Steinhardt, Science **284**, 1481 (1999) [astro-ph/9812133].

¹⁹From the perspective of a dynamical-brane observer, the dilaton field will naturally be viewed as a function of the cosmological time parameter.

- [2] A.J.M. Medved, “A Holographic Interpretation of Asymptotically de Sitter Spacetimes”, hep-th/0112226 (2001).
- [3] R. Bousso, A. Maloney and A. Strominger, “Conformal Vacua and Entropy in de Sitter Space”, hep-th/0112218 (2001).
- [4] M. Spradlin and A. Volovich, “Vacuum States and the S-Matrix in dS/CFT”, hep-th/0112223 (2001).
- [5] R.-G. Cai, “Cardy-Verlinde Formula and Thermodynamics of Black Holes in de Sitter Space”, hep-th/0112253 (2001).
- [6] E. Halyo, “Strings and the Holographic Description of Asymptotically de Sitter Spaces”, hep-th/0201174 (2002).
- [7] Y.S. Myung, “Absorption Cross Section in de Sitter Space”, hep-th/0201176 (2002).
- [8] S. Nojiri and S.D. Odintsov, “Asymptotically de Sitter Dilatonic Space-time, Holographic RG Flow and Conformal Anomaly from (dilatonic) dS/CFT Correspondence”, hep-th/0201210 (2002); S. Nojiri, S.D. Odintsov and S. Ogushi, “Graviton Correlator and Metric Perturbations in de Sitter Brane-world”, hep-th/0202098 (2002).
- [9] S.R. Das, “Thermality in de Sitter and Holography”, hep-th/0202008 (2002).
- [10] M. Brigante, S. Cacciatori, D. Klemm and D. Zanon, “The Asymptotic Dynamics of Two-Dimensional (anti-) de Sitter Gravity”, hep-th/0202073 (2002).
- [11] F. Leblond, D. Marolf and R.C. Myers, “Tall Tales from de Sitter Space I: Renormalization Group Flows”, hep-th/0202094 (2002).
- [12] S. Ness and G. Siopsis, “dS/CFT Correspondence in Two Dimensions”, hep-th/0202096 (2002).
- [13] F. Larsen, J.P. van der Schaar and R.G. Leigh, “De Sitter Holography and the Cosmic Microwave Background”, hep-th/0202127 (2002).
- [14] A. Strominger, JHEP **0110**, 034 (2001) [hep-th/0106113].

- [15] J.M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [hep-th/9711200].
- [16] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Phys. Lett. **B428**, 105 (1998) [hep-th/9802109].
- [17] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150]; *ibid*, 505 (1998) [hep-th/9803131].
- [18] G. 't Hooft, “Dimensional Reduction in Quantum Gravity”, gr-qc/9310026 (1993).
- [19] L. Susskind, J. Math. Phys. **36**, 6377 (1995) [hep-th/9409089].
- [20] See, for instance, M. Spradlin, A. Strominger and A. Volovich, “Les Houches Lectures on De Sitter Space”, hep-th/0110007 (2001).
- [21] R. Bousso, JHEP **0011**, 038 (2000) [hep-th/0010252]; JHEP **0104**, 035 (2001) [hep-th/0012052].
- [22] See, for instance, R.M. Wald, *General Relativity* (University of Chicago Press, 1984).
- [23] V. Balasubramanian, J. de Boer and D. Minic, “Mass, Entropy and Holography in Asymptotically de Sitter Spaces”, hep-th/0110108 (2001).
- [24] V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208**, 413 (1999) [hep-th/9902121].
- [25] J.D. Brown and J.W. York, Phys. Rev. **D47**, 1407 (1993).
- [26] R.-G. Cai, Y. S. Myung and Y.-Z. Zhang, “Check of the Mass Bound Conjecture in de Sitter Space”, hep-th/0110234 (2001).
- [27] A.M. Ghezelbash and R.B. Mann, JHEP **0201**, 005 (2002) [hep-th/0111217].
- [28] D. Birmingham, Class. Quant. Grav. **16**, 1197 (1999) [hep-th/9808032].
- [29] R.-G. Cai, Phys. Lett. **B525**, 331 (2002) [hep-th/0111093].

- [30] A.J.M. Medved, “dS/CFT Duality on the Brane with a Topological Twist” (to appear in Classical and Quantum Gravity), hep-th/0111238 (2001).
- [31] Y.S. Myung, “Dynamic dS/CFT Correspondence using the Brane Cosmology”, hep-th/0112140 (2001).
- [32] U.H. Danielsson, “A Black Hole Hologram in de Sitter Space”, hep-th/0110265 (2001).
- [33] E. Alvarez and C. Gomez, Nucl. Phys. **B541**, 441 (1999) [hep-th/9807226].
- [34] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, Adv. Theor. Math. Phys. **3**, 363 (1999) [hep-th/9904017].
- [35] V. Sahakian, Phys. Rev. **D62**, 126011 (2000) [hep-th/9910099].
- [36] J. de Boer, E. Verlinde and H. Verlinde, JHEP **0008**, 003 (2000) [hep-th/9912012].
- [37] A. Strominger, JHEP **0111**, 049 (2001) [hep-th/0110087].
- [38] L. Susskind and E. Witten, “The Holographic Bound in Anti-de Sitter Space”, hep-th/9805114 (1998).
- [39] A.W. Peet and J. Polchinski, Phys. Rev. **D59**, 065011 (1999) [hep-th/9809022].
- [40] A.B. Zamolodchikov, JETP Lett. **43**, 730 (1986).
- [41] D. Klemm, A.C. Petkou, G. Siopsis and D. Zanon, Nucl. Phys. **B620**, 519 (2002) [hep-th/0104141].
- [42] See, for a general review, M. Cvetič and H.H. Soleng, Phys. Rept. **282**, 159 (1997) [hep-th/9604090].
- [43] H.J. Boonstra, K. Skenderis and P.K. Townsend, JHEP **9901**, 003 (1999) [hep-th/9807137].
- [44] R.-G. Cai and N. Ohta, Phys. Rev. **D62**, 024006 (2000) [hep-th/9912013].

- [45] R.-G. Cai and Y.-Z. Zhang, Phys. Rev. **D64**, 104015 (2001) [hep-th/0105214].
- [46] P.K. Townsend, JHEP **0111**, 042 (2001) [hep-th/0110072].
- [47] K. Behrndt, E. Bergshoeff, R. Halbersma and J.P. Van der Schaar, Class. Quant. Grav. **16**, 3517 (1999) [hep-th/9907006].
- [48] M. Henningson and K. Skenderis, JHEP **9807**, 023 (1998) [hep-th/9806087].
- [49] R. Emparan, C.V. Johnson and R.C. Myers, Phys. Rev. **D60**, 104001 (1999) [hep-th/9903238].
- [50] J.L. Cardy, Nucl. Phys. **B270**, 317 (1986).
- [51] E. Verlinde, “On the Holographic Principle in a Radiation Dominated Universe”, hep-th/0008140 (2000).
- [52] I. Savonije and E. Verlinde, Phys. Lett. **B507**, 305 (2001) [hep-th/0102042].
- [53] J. Jing, Cardy-Verlinde Formula and Entropy Bounds in Kerr-Newman AdS_4/dS_4 Black Hole Backgrounds”, hep-th/0201247 (2002).
- [54] J. Jing, “Cardy-Verlinde Formula and Asymptotically Flat Rotating Charged Black Holes”, hep-th/0202052 (2002).
- [55] G.W. Gibbons and S.W. Hawking, Phys. Rev. **D15**, 2752 (1977).
- [56] G.W. Gibbons and S.W. Hawking, Phys. Rev. **D15**, 2738 (1977).
- [57] R.C. Myers, “Stress Tensors and Casimir Energies in the AdS/CFT Correspondence”, hep-th/9903203 (1999).
- [58] S. Carlip, Phys. Rev. Lett. **82**, 2828 (1999) [hep-th/9812013]; Class. Quant. Grav. **16**, 3327 (1999) [gr-qc/9906126].
- [59] S.N. Solodukhin, Phys. Lett. **B454**, 213 (1999) [hep-th/9812056].
- [60] D. Youm, Mod. Phys. Lett. **A16**, 1327 (2001) [hep-th/0105249].

- [61] E. Halyo, “On the Cardy-Verlinde Formula and the de Sitter/CFT Correspondence”, hep-th/0112093 (2001).
- [62] D. Kutasov and F. Larsen, JHEP **0101**, 001 (2001) [hep-th/0009244].
- [63] F.-L. Lin, Phys. Rev. **D63**, 06402 (2001) [hep-th/0010127].
- [64] I. Brevik, K.A. Milton and S.D. Odintsov, “Entropy Bounds in $R \times S^3$ Geometries”, hep-th/0202048 (2002).
- [65] S.W. Hawking and D.N. Page, Commun. Math. Phys. **87**, 577 (1983).
- [66] S. Ogushi, Mod. Phys. Lett. **A17**, 51 (2002) [hep-th/0111008].
- [67] D. Youm, “A Note on the Cardy-Verlinde Formula”, hep-th/0201268 (2002).